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On a Theorem of E. Egerváry and P. Turán on the Stability of Interpolation

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1. INTRODUCTION

E. Egerváry and P. Turán [3] have introduced the interpolatory polynomial

$$R_n(x,f) = f(1)\frac{1+x}{2}P_n^2(x) + f(-1)\frac{1-x}{2}P_n^2(x) + \sum_{k=1}^{n} f(x_k)\frac{1-x^2}{1-x_k^2}\left(\frac{P_n(x)}{(x-x_k)P_n'(x_k)}\right)^2. \tag{1.1}$$

Here

$$1 = x_0 > x_1 > \cdots > x_n > x_{n+1} = -1 \tag{1.2}$$

are the simple zeros of $(1 - x^2) P_n(x)$ and $P_n(x)$ is the Legendre polynomial of degree n. (By writing $P_n(x)$ rather than $P_{n-2}(x)$ in (1.1) we have made a slight change in the notation used in Ref. 3.) This process satisfies

$$R_n(x_i, f) = f(x_i) \qquad \text{for} \quad i = 0, 1,..., n+1$$

and

$$R_n'(x_i, f) = 0 \qquad \text{for} \quad i = 1, 2,..., n$$

and is the most economical stable interpolation process. Recently S. Karlin [4, Chap. 10] has given an extensive discussion of this and similar processes and their relevance to the theory of experimental design.

THEOREM 1. (E. Egerváry and P. Turán.) *The sequence of interpolatory polynomials $R_n(x,f)$ converges uniformly in $[-1, 1]$ to $f(x)$ as $n \to \infty$ whenever $f(x)$ is continuous in $-1 \leqslant x \leqslant 1$.*

The aim of this paper is to determine the rate of convergence of $R_n(x,f)$ in terms of the arithmetic means of the sequence $(\omega_f(1/n))$. Essential to our proof is an important lemma of R. Bojanic [2].

Let $\Omega(t)$ be an increasing, subadditive, continuous function defined for non-negative values of t such that $\Omega(0) = 0$. Let $C_M(\Omega)$ be the class of continuous functions on $[-1, 1]$ defined by $f \in C_M(\Omega)$ if and only if $\omega_f(h) \leqslant M\Omega(h)$ for all $h \geqslant 0$. Equivalently, $f \in C_M(\Omega)$ if and only if

$$|f(x) - f(y)| \leqslant M\Omega(|x - y|) \tag{1.2}$$

for all $x, y \in [-1, 1]$.

We shall prove the following.

THEOREM 2. *There exist constants c_1 and c_2 $(0 < c_1 < c_2 < \infty)$ such that for $n \geqslant 5$,*

$$\frac{c_1 M}{n} \sum_{r=5}^{n} \Omega\left(\frac{1}{r}\right) \leqslant \sup_{f \in C_M(\Omega)} \| R_n(f) - f \| \leqslant \frac{c_2 M}{n} \sum_{r=1}^{n} \Omega\left(\frac{1}{r}\right). \tag{1.3}$$

2. Preliminaries

It is well known that

$$\Omega(\lambda t) \leqslant (\lambda + 1)\, \Omega(t) \qquad \text{for all} \quad \lambda \geqslant 0, \quad t \geqslant 0, \tag{2.1}$$

$$0 < t_1 < t_2 \Rightarrow 2\,\frac{\Omega(t_1)}{t_1} \geqslant \frac{\Omega(t_2)}{t_2}. \tag{2.2}$$

We shall also need the following results concerning Legendre polynomials. For $-1 \leqslant x \leqslant 1$ and $n = 1, 2, 3,\ldots$, [3]

$$\sum_{k=1}^{n} \frac{1 - x^2}{1 - x_k^2} \left(\frac{P_n(x)}{(x - x_k)\, P_n'(x_k)}\right)^2 \equiv 1 - P_n^2(x) \leqslant 1, \tag{2.3}$$

$$(1 - x^2)^{1/4}\, |P_n(x)| < \sqrt{\frac{2}{\pi n}}, \tag{2.4}$$

and

$$(1 - x^2)(P_n'(x))^2 \leqslant n^2. \tag{2.5}$$

Recalling the definition of the x_k's, we have (see Ref. 5)

$$|P_n'(x_k)| \sim k^{-3/2} n^2, \qquad k = 1, 2,\ldots, \left[\frac{n}{2}\right], \tag{2.6}$$

$$|P_n'(x_k)| \sim (n + 1 - k)^{-3/2}\, n^2, \qquad k = \left[\frac{n}{2}\right] + 1,\ldots, n, \tag{2.7}$$

$$(1 - x_k^2) > \frac{(k - 1/2)^2}{(n + 1/2)^2} \qquad k = 1, 2,\ldots, \left[\frac{n}{2}\right], \tag{2.8}$$

$$(1 - x_k^2) > \frac{(n - k + 1/2)^2}{(n + 1/2)^2} \qquad k = \left[\frac{n}{2}\right] + 1,\ldots, n, \tag{2.9}$$

$$\frac{(k - 1/2)\pi}{(n + 1/2)} < \theta_k < \frac{k\pi}{(n + 1/2)} \qquad k = 1, 2,\ldots, n, \quad x_k = \cos\theta_k. \tag{2.10}$$

3. Estimates

Lemma 1. (R. Bojanic [2].) *Let $\Omega(t)$ be a modulus of continuity. Then, for $m \geqslant s \geqslant 2$, we have*

$$\frac{\pi}{m}\int_{\pi/m}^{s\pi/m}\frac{\Omega(t)}{t^2}\,dt \leqslant \sum_{r=1}^{s-1}\frac{1}{r^2}\,\Omega\left(\frac{(r+1)\pi}{m}\right) \leqslant \frac{8\pi}{m}\int_{\pi/m}^{s\pi/m}\frac{\Omega(t)}{t^2}\,dt. \tag{3.1}$$

Lemma 2. *Let $-1 \leqslant x \leqslant +1$ and let x_j be that zero of $P_n(x)$ which is nearest to x. Then we have*

$$\frac{(1-x^2)\,P_n^2(x)}{(1-x_k^2)(P_n'(x_k))^2\,(x-x_k)^2} \leqslant \frac{c}{r^2} \qquad \text{for} \quad k=j\pm r, \quad 1 \leqslant r \leqslant n. \tag{3.2}$$

Proof. Let $x = \cos\theta$, $x_k = \cos\theta_k$. By using the definition of j and (2.10) it follows that

$$\frac{1}{\sin^2\dfrac{\theta-\theta_k}{2}} \leqslant \frac{(n+1/2)^2}{(r-1/2)^2}.$$

Since

$$\sin\theta \leqslant \sin\theta + \sin\theta_k \leqslant 2\sin\frac{\theta+\theta_k}{2},$$

and

$$\sin\theta_k \leqslant \sin\theta_k + \sin\theta \leqslant 2\sin\frac{\theta+\theta_k}{2},$$

we have, on using (2.8)

$$\frac{\sin\theta}{(\cos\theta-\cos\theta_k)^2} = \frac{1}{4\sin^2\dfrac{\theta-\theta_k}{2}}\;\frac{\sin\theta}{\sin^2\dfrac{\theta+\theta_k}{2}},$$

$$\leqslant \frac{(n+1/2)^2}{(r-1/2)^2}\,\frac{1}{\sin\theta_k} \leqslant \frac{(n+1/2)^3}{(r-1/2)^2\,(k-1/2)}. \tag{3.3}$$

On using (2.4) and (2.6–2.9) we have

$$\frac{(1-x^2)^{1/2}\,P_n^2(x)}{(1-x_k^2)(P_n'(x_k))^2} \leqslant \frac{2}{\pi n}\,\frac{k}{n^2} = \frac{2k}{\pi n^3}.$$

Now on using (3.3) and the above result we have (3.2).

LEMMA 3. (J. Balázs and P. Turán.) *For the relative extrema of* $P_{n-1}(x)$ *in* $0 < x < 1$ *we have the estimation* ($n = 4, 6, 8, \ldots$)

$$
\begin{aligned}
| P_{n-1}(\xi_\nu)| &\geqslant \frac{1}{\sqrt{8\pi\nu}}, \qquad && \nu = 2, 3, \ldots, \left[\frac{n}{2}\right] \\
&\geqslant \frac{1}{\sqrt{8\pi(n-\nu)}}, \qquad && \nu = \left[\frac{n}{2}\right] + 1, \ldots, n-1.
\end{aligned}
$$

Furthermore, a similar estimate holds for odd values of n.

For the proof we refer to Lemma 2.1 and corresponding remarks at the end of page 202 in Ref. 1. In our case we need the above lemma for one particular value of ξ_ν. Let ξ be the first positive zero of $P_n'(x)$ in $0 < x < 1$. Then from above it follows that

$$
| P_n(\xi)| \geqslant \frac{c_4}{\sqrt{n}} \quad \text{with} \quad c_4 = \frac{1}{2\sqrt{\pi}}. \tag{3.4}
$$

LEMMA 4. *Let* ξ *be the first positive zero of* $P_n'(x)$ *in* $(0, 1)$. *Then, for* $n \geqslant 5$,

$$
\begin{aligned}
A &\equiv \sum_{k=1}^{n} \frac{1-\xi^2}{1-x_k^2} \left(\frac{P_n(\xi)}{(\xi - x_k) P_n'(x_k)} \right)^2 \Omega(| \xi - x_k |) \\
&\geqslant \frac{c_2}{n} \sum_{r=5}^{n} \Omega\left(\frac{1}{r}\right).
\end{aligned}
$$

Proof. Let $\xi = \cos\eta$. Then for $k \leqslant [n/2] - 1$ it follows from (2.10) that

$$
\begin{aligned}
| \xi - x_k | &\leqslant \eta - \theta_k \leqslant \pi/2 - \theta_k \\
&\leqslant \frac{\left(\frac{n}{2} - k + \frac{3}{4}\right)\pi}{n}.
\end{aligned} \tag{3.5}
$$

Hence, by (2.2), (2.6), (2.10) and (3.5),

$$
\begin{aligned}
A &\geqslant \frac{c_5}{n} \sum_{k=1}^{[n/2]-1} \frac{k}{\left(\frac{n}{2} - k + \frac{3}{4}\right)^2} \Omega\left(\frac{\left(\frac{n}{2} - k + \frac{3}{4}\right)\pi}{n} \right) \\
&\geqslant c_6 \sum_{k=[n/4]}^{[n/2]-1} \frac{\Omega\left(\frac{\left(\frac{n}{2} - k + \frac{3}{4}\right)\pi}{n} \right)}{\left(\frac{n}{2} - k + \frac{3}{4}\right)^2}.
\end{aligned}
$$

Now by writing

$$\left[\frac{n}{2}\right] - k = r$$

and using the properties of Ω we have

$$A \geqslant c_7 \sum_{r=1}^{[n/2]-[n/4]} \frac{\Omega\left(\frac{(j+1)\pi}{n}\right)}{j^2}. \tag{3.6}$$

We now let $m = n$, $s = [n/2] - [n/4] + 1$ and apply Lemma 1 to (3.6) observing that $(s/n) > (1/4)$. Hence,

$$\begin{aligned} A &\geqslant \frac{c_8}{n} \int_{\pi/n}^{s\pi/n} \frac{\Omega(t)}{t^2}\, dt \\ &\geqslant \frac{c_8}{n} \int_{\pi/n}^{\pi/4} \frac{\Omega(t)}{t^2}\, dt \\ &= \frac{c_2}{n} \int_4^n \Omega\left(\frac{\pi}{t}\right) dt \\ &\geqslant \frac{c_2}{n} \int_4^n \Omega\left(\frac{1}{t}\right) dt \\ &\geqslant \frac{c_2}{n} \sum_{r=5}^{n} \Omega\left(\frac{1}{r}\right). \end{aligned}$$

4. Proof of Theorem 2.

Let

$$f \in C_M(\Omega). \tag{4.1}$$

From (1.1) and (2.3) we have

$$\begin{aligned} &| R_n(x, f) - f(x)| \\ &\leqslant \left\{\frac{(1+x)}{2} |f(1) - f(x)| + \frac{(1-x)}{2} |f(-1) - f(x)|\right\} P_n^2(x) \\ &\quad + \sum_{k=1}^{n} \frac{1-x^2}{1-x_k^2} \left(\frac{P_n(x)}{P_n'(x_k)(x-x_k)}\right)^2 |f(x_k) - f(x)| \\ &= S_1 + S_2 . \end{aligned} \tag{4.2}$$

We shall estimate each of the sums S_1 and S_2 separately. By (4.1), (2.1), (2.3), (2.4) and the monotonicity of Ω

$$S_1 \leqslant M\left[\frac{(1+x)}{2}\,\Omega(1-x)+\frac{(1-x)}{2}\,\Omega(1+x)\right]P_n{}^2(x)$$

$$\leqslant M\Omega\left(\frac{1}{n}\right)[n(1-x^2)\,P_n{}^2(x)+P_n{}^2(x)]$$

$$\leqslant M\left(\frac{2+\pi}{\pi}\right)\Omega\left(\frac{1}{n}\right).$$

Hence,

$$S_1 \leqslant \frac{2M}{n}\sum_{r=1}^{n}\Omega\left(\frac{1}{r}\right). \tag{4.3}$$

Choose j such that $1 \leqslant j \leqslant n$ and $|x - x_j| \leqslant |x - x_k|$, $k = 1, 2,..., n$. Now, since $f \in C_M(\Omega)$,

$$S_2 \leqslant M\sum_{k=1}^{n}\frac{1-x^2}{1-x_k{}^2}\left(\frac{P_n(x)}{P_n{}'(x_k)(x-x_k)}\right)^2\Omega(|x_k - x|)$$

$$S_2 \leqslant M{\sum_{k=1}^{n}}' + MU_j \tag{4.4}$$

where $\sum_{k=1}^{\prime n}$ signifies that the jth term, U_j, has been omitted.

Before estimating $\sum_{k=1}^{\prime n}$, let us note that for $k = j \pm r$, $r \geqslant 1$,

$$\Omega(|x_k - x|) = \Omega(|\cos\theta_k - \cos\theta|)$$
$$\leqslant \Omega(|\theta_k - \theta|).$$

Thus, by (2.10), and for $k = j \pm r$

$$\Omega(|x_k - x|) \leqslant c_9\Omega\left(\frac{(r+1/2)\pi}{n+1/2}\right). \tag{4.5}$$

Hence by (3.2) and (4.5) and the property (2.1) of Ω,

$$M{\sum_{k=1}^{n}}' \leqslant Mc_{10}\sum_{r=1}^{n-1}\frac{1}{r^2}\,\Omega\left(\frac{(r+1)\pi}{2n}\right). \tag{4.6}$$

Let us choose $m = 2n$ and $s = n$ in Lemma 1 and apply it to (4.3). Then we obtain,

$$M \sum_{k=1}^{n}{}' \leqslant \frac{8\pi c_{10} M}{n} \int_{\pi/2n}^{\pi} \frac{\Omega(t)}{t^2}\, dt$$

$$\leqslant \frac{c_{10} M}{n} \int_{1}^{2n} \Omega\left(\frac{\pi}{t}\right) dt$$

$$\leqslant \frac{4 c_{10} M}{n} \int_{1}^{2n} \Omega\left(\frac{1}{t}\right) dt.$$

Hence

$$M \sum_{k=1}^{n}{}' \leqslant \frac{c_{11}}{n} \sum_{r=1}^{n} \Omega\left(\frac{1}{r}\right). \tag{4.7}$$

We now turn to estimating U_j. By (2.10), (2.3), and the monotonicity of Ω, it follows that

$$MU_j = M \frac{1 - x^2}{1 - x_j^2} \left(\frac{P_n(x)}{(x - x_j)\, P_n'(x_j)}\right)^2 \Omega(|\, x - x_j\,|)$$

$$\leqslant M\Omega(|\, \theta - \theta_j\,|).$$

Since $|\, \theta - \theta_j\,| < 2/n$ we have

$$MU_j \leqslant 2M\Omega\left(\frac{1}{n}\right) \leqslant \frac{2M}{n} \sum_{r=1}^{n} \Omega\left(\frac{1}{r}\right). \tag{4.8}$$

Thus from (4.2), (4.3), (4.4), (4.6), and (4.8), the upper estimate in Theorem 2 follows.

To prove the lower estimate in Theorem 2, consider the function

$$g(x) = M\Omega(|\, x - \xi\,|)$$

where ξ is as in Lemma 4. It is easily verified that $g \in C_M(\Omega)$.

Hence

$$\sup_{f \in C_M(\Omega)} \|\, R_n(f) - f \,\| \geqslant \|\, R_n(g) - g \,\|$$

$$\geqslant |\, R_n(g, \xi) - g(\xi)|$$

$$= R_n(g, \xi).$$

The lower estimate in Theorem 2 now follows from Lemma 4.

References

1. J. Balázs and P. Turán, Notes on interpolation III (Convergence), *Acta Math. Acad. Sci. Hung.* **9** (1958), 195–214.
2. R. Bojanic, A note on the precision of interpolation by Hermite-Fejér polynomials, *Proceedings of the conference on Constructive Theory of function (Budapest)* 1971, 69–76.
3. E. Egerváry and P. Turán, Notes on interpolation V (On the stability of interpolation), *Acta Math. Acad. Sci. Hung.* **9** (1958), 259–267.
4. S. Karlin and W. J. Studden, "Tchebycheff Systems," Interscience Publications, New York, 1966.
5. G. Szegö, Orthogonal Polynomials, *Amer. Math. Soc. Coll. Publ.* **23** (1967).